INTERIOR POINTS OF CONVEX HULLS

BY

WILLIAM E. BONNICE AND JOHN R. REAY

ABSTRACT

If a set $X \subset E^n$ has non-empty k-dimensional interior, or if some point is k-dimensional surrounded, then the classic theorem of E. Steinitz may be extended. For example if $X \subset E^n$ has $\operatorname{int}_k X \neq 0$, $(0 \leqq k \leqq n)$ and if $p \in \operatorname{int} \operatorname{con} X$, then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with card $Y \leqq 2n - k + 1$.

1. **Results.** For X in a linear space, the *k-interior* of X, denoted $\operatorname{int}_k X$, is the set of all points s such that s is in the relative interior of some k-simplex contained in X; equivalently $s \in \operatorname{int}_k X$ if and only if there exists a k-dimensional flat F such that s is interior to $X \cap F$ relative to F. Note that $\operatorname{int}_0 X = X$ and if X is a subset of an n-dimensional space then $\operatorname{int}_n X$ is the usual interior of X, which will also be denoted by $\operatorname{int} X$. We will let $\operatorname{con} X$, $\operatorname{aff} X$, $\operatorname{lin} X$, and $\operatorname{card} X$ denote the convex hull, the affine span, the linear span, and the cardinality of X, respectively. Of the following three results, the first two are due to Steinitz [5] and the third to Reay [4].

A. If $X \subset E^n$ and $p \in \operatorname{int} \operatorname{con} X$ then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y \leqq 2n$.

B. If $X \subset E^n$ is not contained in the union of n lines through p and if $p \in \operatorname{int} \operatorname{con} X$ then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y \leqq 2n - 1$.

C. For $n \geqq 3$, if $X \subset E^n$ is connected and if $p \notin X$ but $p \in \operatorname{int} \operatorname{con} X$, then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y \leqq 2n - 2$.

The purpose of this paper is to prove (with other conditions on X) some analogous results, principally the following:

1.1 THEOREM. *If $X \subset E^n$ has $\operatorname{int}_k X \neq \varnothing$ $(0 \leqq k \leqq n)$ and if $p \in \operatorname{int} \operatorname{con} X$ then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y \leqq 2n - k + 1$.*

1.2 THEOREM. *For $X \subset E^n$ if there is a k-dimensional flat F $(0 \leqq k \leqq n-1)$ with $\operatorname{int}_k (X \cap F) \neq \varnothing$ (thus $\operatorname{int}_k X \neq \varnothing$), and if $p \notin F$ is such that $p \in \operatorname{int} \operatorname{con} X$, then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y \leqq 2n - k$.*

1.3 COROLLARY. *If $X \subset E^n$ has $\operatorname{int} X \neq \varnothing$ and if $p \in \operatorname{int} \operatorname{con} X$ then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with $\operatorname{card} Y = n + 1$.*

Note that 1.3 is a special case of both 1.1 (with $k = n$) and of 1.2 (with

Received October 26, 1966

$k = n - 1$). Except for the case $k = 0$ in 1.1, each of these is a best possible result in the sense that for each result there are configurations which satisfy the hypothesis, but for which card Y cannot be reduced further.

1.4 EXAMPLE In 1.1, to see that $2n - k + 1$ is "best" for $1 \leqq k \leqq n$ let p be the origin in E^n, let B be a solid k-dimensional ball centered at p, let A be a linear basis for an $(n-k)$-dimensional linear subspace supplementary to $\operatorname{lin} B$ ($A = \emptyset$ when $k = n$) and let $X = B \cup A \cup (-A) = \{-a \mid a \in A\}$. Then if $p \in \operatorname{int} \operatorname{con} Y$ with $X \subset X$, Y must contain $A \cup (-A)$ together with at least $k+1$ points of B, so card $Y \geqq 2(n-k) + k + 1 = 2n - k + 1$.

1.5 EXAMPLE. In 1.2, to see that $2n - k$ is "best" for $0 \leqq k \leqq n-1$, let p be the origin in E^n, let $q \neq p$, and let T be a solid k-dimensional ball centered at $-q$. with $p \notin \operatorname{aff} T$. (For $k = 0$, let $T = \{-q\}$.) Let V be a linear basis for an $(n-k-1)$-dimensional linear subspace supplementary to $\operatorname{lin} T = \operatorname{aff}(\{p\} \cup T)$ and let $X = \{q\} \cup T \cup V \cup (-V)$. Then if $p \in \operatorname{int} \operatorname{con} Y$ with $Y \subset X$, Y must contain $V \cup (-V) \cup \{q\}$ together with at least $k+1$ points of T, so card $Y \geqq 2(n-k-1) + k + 2 = 2n - k$.

2. **Proofs.** The proofs will be based on positive bases. Early papers on pos tive bases were those of Chandler Davis [2] and McKinney [3]. We will use the terminology and theory as presented in Bonnice-Klee [1, pp. 5–7], and Reay [4, pp. 5–8]. In brief, for a set U contained in a linear space L, pos U will denote the set of all finite linear combinations of elements of U having all coefficients nonnegative. If A is a subset of L such that pos $U = A$ then we say that U *positively spans* A. Thus pos U is the cone or "wedge" generated by U having the origin as vertex. The set U is said to be *positively independent* if for all $u \in U$, $u \notin \operatorname{pos}(U \sim \{u\})$. If U is positively independent and positively spans a linear space L, then U is a *positive basis* for L. Every linear space L admits a positive basis. In fact, if L is n-dimensional and k is the cardinality of a positive basis for L then $n + 1 \leqq k \leqq 2n$ and moreover all of these values of k are realizable. A basis with cardinality $n+1$ is called a *minimal basis* for L. A linear subspace S of L is a *spanned subspace with respect to* U if $U \cap S$ *positively spans* S. *In this case, if* S is k-dimensional ($k \geqq 1$) and $U \cap S$ *has* $k+1$ members (and hence $U \cap S$ is a minimal basis for S), S is called a *minimal subspace with respect to* U. The important connecting link between positive bases and the results we want to derive about a point $p \in \operatorname{int} \operatorname{con} X$ is the obvious fact that for X contained in an n-dimensional space E, $0 \in \operatorname{int} \operatorname{con} X$ if and only if pos $X = E$.

2.1 LEMMA. *If X positively spans an n-dimensional linear space E, if M is a d-dimensional $(1 \leqq d \leqq n)$ subspace of E and if $U \subset X$ is such that* pos $U = M$ *where* card $U = f$, *then there is a subset Y of X with card $Y \leqq 2(n-d) + f$ which positively spans E, and hence the origin is in* int con Y. *In particular if M is a minimal subspace with respect to X,* card $Y \leqq 2n - d + 1$.

Proof. The case $d = n$ has been noted above. For $1 \leqq d \leqq n-1$, let S be an $(n-d)$-dimensional subspace of E such that E is the direct linear sum of M and S and let π be the linear projection of E onto S with kernel M. Then $\pi(X \sim M)$ positively spans S and hence there is a subset W of $X \sim M$ with $n-d+1 \leqq \operatorname{card} W \leqq 2(n-d)$ such that πW positively spans S. By assumption there is a $U \subset X$ with card $U = f$ such that pos $U = M$. Letting $Y = U \cup W$, it follows that pos $Y = E$ (For details see [1, Lemma 2.7] or [4, Lemma 2.3]) and that card $Y \leqq 2(n-d)+f$.

Proof of 1.1 and 1.2. Since we may assume that p is the origin, 1.1 follows from 2.1 once it is shown that there is a minimal subspace with respect to X having dimension at least k. To do this, begin by taking a point $q \in \operatorname{int}_k X$ such that $q \neq p$ and let B^k be a closed solid k-dimensional ball centered at q and contained in X. By A. above, p is interior to some subset of X which has at most $2n$ points. Adjoining this finite set to B^k we obtain a compact subset having p in its interior. Thus we may assume that X is compact and hence so is con X. Then the ray from q through p intersects the boundary of con X in some point $r \in \operatorname{con} X$. Let H be a hyperplane through r supporting con X. Since $H \cap \operatorname{con} X = \operatorname{con}(H \cap X$ by a theorem of Caratheodory ([4, Theorem 1.1]) there is a subset T of $H \cap X$ with card $T \leqq n$ such that $r \in \operatorname{con} T$. By taking card T minimal with respect to the property that $r \in \operatorname{con} T$, T may be assumed to consist of the verticles of a j-simplex $(0 \leqq j \leqq n-1)$ having r in its relative interior. Let L denote pos $(\{q\} \cup T)$ and let F denote aff B^k, so that $L \cap F$ is an affine space containing q. If $L \cap F = F$ then $L \supset F$ and since q is in X, L is a minimal subspace with respect to X having sufficient dimension. If $L \cap F$ is properly contained in F, let A be an affine subspace of F which would be supplementary to $L \cap F$ in F if q were the origin. Letting e denote the dimension of A, then $e \geqq 1$ and $A \cap B^k = B^e$ is a closed e-dimensional ball centered at q. Thus there is an e-simplex with vertex set V contained in B^e and having q in its relative interior. Then $\operatorname{pos}(V \cup T) = \operatorname{lin}(V \cup T)$ and

$$\dim \operatorname{pos}(V \cup T) = \dim \operatorname{aff} T + 1 = (\operatorname{card} V) - 1 + \operatorname{card} T$$

so $\operatorname{pos}(V \cup T)$ is a minimal subspace with respect to X which contains F (and p) and so is at least k-dimensional. This completes the proof of 1.1.

To prove 1.2, we note that under the stronger hypothesis of 1.2, B^k may be chosen in the above proof so that $p \notin F = \operatorname{aff} B^k$ and hence the ray from q through p intersects F only at q. Therefore $\operatorname{aff}(\{p\} \cup F)$ is $(k+1)$-dimensional. But minimal subspace $\operatorname{pos}(V \cup T)$ contains $\operatorname{aff}(\{p\} \cup F)$ and therefore is at least $(k+1)$-dimensional. Now by 2.1 there is a subset Y of X with $\operatorname{card} Y \leqq 2n - (k+1) + 1 = 2n - k$ such that $p \in \operatorname{int} \operatorname{con} Y$.

3. **Apparent interiors.** The contribution of a point $x \in X$ to the set pos X is not determined by how far x is from the origin, but only by the direction of x

from the origin. That is, x may be replaced in X by αx (where α is any positive number) and pos X will not be changed. As a result conditions of the form "X is connected" or "$\mathrm{int}_k\, X \neq \emptyset$" may often be replaced by much weaker conditions in theorems where the proofs use the theory of positive bases.

As an example, for a given point $p \in E^n$, let π_p denote the usual radial projection of $E^n \sim \{p\}$ onto the unit sphere centered at p. We say that X *appears to be connected from* p provided that $\pi_p(X \sim \{p\})$ is a connected subset of the sphere. The result C in the first section may now be given in the much stronger form:

D. For $n \geqq 3$, if $X \subset E^n$ appears to be connected from p and $p \in \mathrm{int\,con}\, X$ then $p \in \mathrm{int\,con}\, Y$ for some $Y \subset X$ with card $Y \leqq 2n-2$.

Note that the set X may appear connected from p and yet be totally disconnected.

To obtain a similar generalization of 1.1 and 1.2, we say that X *appears to have k-interior from p* if there exists a k-dimensional flat F missing p and having the following property:

If F' denotes the cone $p + \mathrm{pos}(F - p)$, (that is, the cone with vertex p and generated by F) and if π_p maps F' radially from p onto F then $\mathrm{int}_k\, \pi_p(X \cup F') \neq \emptyset$. Equivalently, X *appears to have k-interior from p* if there exists a k-dimensional flat F missing p and such that in $L = \mathrm{aff}(\{p\} \cup F)$ if F' denotes the open "half of L" containing F and bounded by the translation of F to p then the radial proection of $X \cup F'$ from p has nonempty k-interior.

3.1 THEOREM. *If $X \subset E^n$ appears to have k-interior from $p(0 \leqq k \leqq n-1)$, and if $p \in \mathrm{int\,con}\, X$, then $p \in \mathrm{int\,con}\, Y$ for some $Y \subset X$ with* card $Y \leqq 2n-k$.

Proof. We may assume that p is the origin. With the notation of the preceeding definition, let $X' = \pi_p(X \cap \mathrm{pos}\, F) \subset F$ and apply 1.2 to get a subset Y' of $X \cup X'$ for which $p \in \mathrm{int\,con}\, Y'$ with card $Y' \leqq 2n-k$. Now for each $y \in Y'$, $\alpha_y y \in X$ for some $\alpha_y > 0$. Let $Y = \{a_y \alpha \mid y \in Y'\}$. Then $Y \subset X$, $p \in \mathrm{int\,con}\, Y$, and card $Y \leqq 2n-k$.

For $1 \leqq k \leqq n$, we say that $q \in E^n$ is *k-surrounded* by $X \subset E^n$ if there exists a k-dimensional flat F through q with the following property: If π_q is the radial projection of E^n onto the unit n-shell $\{z \in E : |z-q| = 1\}$ centered at q, then $\pi_q(X \cap F)$ is all of the unit k-shell at q in F. (Note that if q is k-surrounded, then X appears to have $(k-1)$-interior from q.) With this notation we may obtain a theorem concerning any $p \in \mathrm{int\,con}\, X$ whether X appears to have k-interior from p or not.

3.2 THEOREM. *For $X \subset E^n$ if there exists some point of E^n which is k-surrounded* $(1 \leqq k \leqq n)$ *by X and if $p \in \mathrm{int\,con}\, X$ then $p \in \mathrm{int\,con}\, Y$ for some $Y \subset X$ with* card $Y \leqq 2n-k+1$.

That $2n-k+1$ is "best" is seen from Example 1.4.

3.3 THEOREM. *For $X \subset E^n$ and $p \in \operatorname{int} \operatorname{con} X$ if there is a k-dimensional flat F missing p $(1 \leqq k \leqq n-1)$ and a point $q \in F$ for which $\pi_q(X \cap F)$ is the entire unit k-shell at q in F (thus q is k-surrounded by X) then $p \in \operatorname{int} \operatorname{con} Y$ for some $Y \subset X$ with* $\operatorname{card} Y \leqq 2n - k + 1$. *Moreover, if there is a such a q in X then there is a Y as stated but having* $\operatorname{card} Y \leqq 2n - k$.

That $2n - k$ is best seen from Example 1.5. The following example shows that $2n - k + 1$ cannot be improved on.

3.4 EXAMPLE. Let $V \cup W$ be a linear basis for E^n where V and W are disjoint and V has $k + 1$ members. Let $X = (-W) \cup W \cup (-V) \cup$ boundary con V. Then $F = \operatorname{aff} V$ is a k-dimensional flat and, if q is in the relative interior of k-simplex con V, $\pi_q(X \cup F)$ is the entire k-shell at q in F. Letting p be the origin, if $p \in \operatorname{int}$ con Y with $Y \subset X$, Y must contain $(-W) \cup W \cup (-V)$ together with at least two points of the boundary of con V. Thus $\operatorname{card} Y \geqq 2(n - k - 1) + (k + 1) + 2 = 2n - k + 1$.

Proof of 3.2 **and 3.3.** We may assume p is the origin. Since some q is k-surrounded by X, there is a k-dimensional flat F through q for which $\pi_q(X \cap F)$ is all of the unit k-shall S at q in F. We may assume that $q \neq p$ because 3.2 follows from 3.1 in the special case when $p = q$. The proofs proceed as the proofs of 1.1 and 1.2 except that the role of B^k (the closed solid k-dimensional ball centered at q) is played by S. So in this setting, F denotes aff$S$ and again L denotes $\operatorname{pos}([q] \cup T) = \operatorname{aff}(\{p\} \cup T)$. If q is in X or if $L \cap F$ is a proper affine subspace of F then as in that proof we obtain a subset V of S whose convex hull is an e-simplex having q in its relative interior, and $V \cup T$ is a minimal positive basis for the subspace $\operatorname{pos}(V \cup T)$ of dimension at least k (dimension at least $k + 1$ with the hypothesis of 3.3). Now a point $v \in V$ might not be an element of X, but it is the radial projection $\pi_q x_v$ of some point $x_v \in X$. If V' is the set of $e + 1$ points $x_v \in X$ thus obtained from the $e + 1$ points of V, then pos $(V' \cup T) =$ pos $(V \cup T$ and the proof is completed by applying 2.1 as before, obtaining $Y \subset X$ with $p \in$ int con Y and card $Y \leqq 2n - k + 1$ in case of 3.2 and card $Y \leqq 2n - k$ in the case of 3.3.

There remains only the case where $q \notin X$. In this case for both 3.2 and 3.3, we have to find a $Y \subset X$ with $p \in \operatorname{int} \operatorname{con} Y$ and $\operatorname{card} Y \leqq 2n - k + 1$. As noted above, we may assume that $L \cap F = F$ and hence $L \supset F$.

If $p \in F$, then ray pq is contained in F and therefore intersects S. Thus there is a point $x \in X$ on the open ray from p through q. Hence $\operatorname{pos}(\{x\} \cup T)$ equals L and is a minimal subspace with respect to X having dimension $\geqq k$. Again 2.1 yields a desired Y.

So assume that $p \notin F$. Any line in F through q has at least one point of X on each side of q. Let x_1 and x_2 be two such points of X such that $q \in \operatorname{con}\{x_1, x_2\}$. Then $\operatorname{pos}(\{x_1, x_2\} \cup T) \supset \operatorname{pos}(\{q\} \cup T) = L$ and, since x_1 and x_2 are in

$F \subset L$, pos $(\{x_1,x_2\} \cup T) = L$. With $j+1$ denoting card T, since $p \in L \sim F$, $j+1 = \dim L \geqq \dim F + 1 = k+1$. Applying 2.1 with $j+1$ playing the role of d and $f = \text{card}(\{x_1,x_2\} \cup T)$ (thus $j+1 \leqq f \leqq j+3$) there is a $Y \subset X$ such that $p \in \text{int con } Y$ and with card

$$Y \leqq 2[n-(j+1)] + f \leqq 2n - j + 1 \leqq 2n - k + 1 .$$

References

1. Wm. Bonnice and V. L. Klee, *The generation of convex hulls*, Math. Ann. **152** (1963), 1–29.

2. Chandler Davis, *Theory of positive linear dependence*, Amer. J. Math. **76** (1954), 733–746·

3. R. L. McKinney, *Positive bases for linear spaces*, Trans. Amer. Math. Soc. **103** (1962), 131–148.

4. J. R. Reay, *Generalization of a theorem of Carathéodory*, Amer. Math. Soc. Memoir No. 54, 1965.

5. E. Steinitz, *Bedingt konvergente Reihen und konvexe Systeme I-II-III*, J. Reine Angew. Math. **143** (1913), 128–175, **144** (1914), 1–40, **146** (1916), 1–52.

UNIVERSITY OF NEW HAMPSHIRE,
DURHAM, NEW HAMPSHIRE,
MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICHIGAN
AND
WESTERN WASHINGTON STATE COLLEGE